Math 4550
Homework 5
Solutions

1 $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

$\mathbb{Z}_8$ is cyclic so all its subgroups are cyclic.

$\langle \bar{0} \rangle = \{\bar{0}\}$

$\langle \bar{1} \rangle = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\} = \mathbb{Z}_8$

$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$

$\langle \bar{3} \rangle = \{\bar{0}, \bar{3}, \bar{6}, \bar{1}, \bar{4}, \bar{7}, \bar{2}, \bar{5}\} = \mathbb{Z}_8$

$\langle \bar{4} \rangle = \{\bar{0}, \bar{4}\}$

$\langle \bar{5} \rangle = \{\bar{0}, \bar{5}, \bar{2}, \bar{7}, \bar{4}, \bar{1}, \bar{6}, \bar{3}\}$

$\langle \bar{6} \rangle = \{\bar{0}, \bar{6}, \bar{4}, \bar{2}\}$

$\langle \bar{7} \rangle = \{\bar{0}, \bar{7}, \bar{6}, \bar{5}, \bar{4}, \bar{3}, \bar{2}, \bar{1}\} = \mathbb{Z}_8$

subgroups

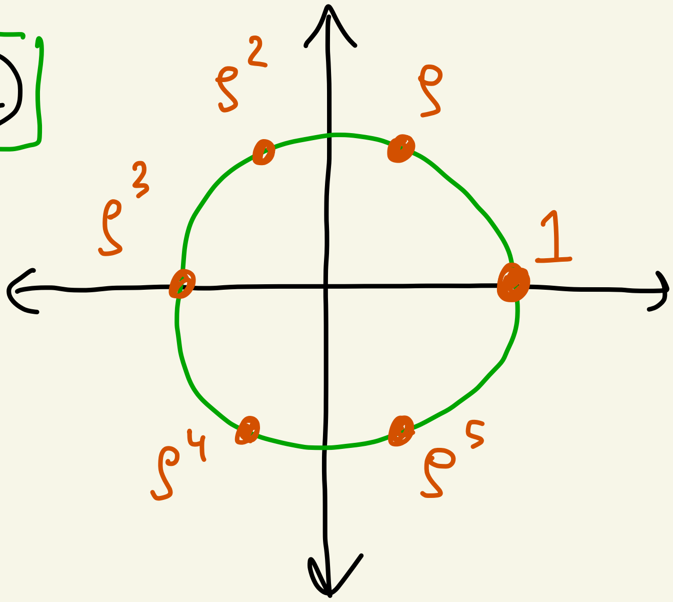
$\{\bar{0}\}$

$\{\bar{0}, \bar{4}\}$

$\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$

$\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

(2)



$U_6 = \{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$

where $\rho = e^{\frac{2\pi i}{6}}$

and $\rho^6 = 1$

U_6 is cyclic so all its subgroups are cyclic.

$\langle 1 \rangle = \{1\}$

$\langle \rho \rangle = \{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$

$\langle \rho^2 \rangle = \{1, \rho^2, \rho^4\}$

$\langle \rho^3 \rangle = \{1, \rho^3\}$

$\langle \rho^4 \rangle = \{1, \rho^4, \rho^2\}$

$\langle \rho^5 \rangle = \{1, \rho^5, \rho^4, \rho^3, \rho^2, \rho\}$

Subgroups:

$\{1\}$

$\{1, \rho^3\}$

$\{1, \rho^2, \rho^4\}$

$\{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$

③

$$U_5 = \{1, \rho, \rho^2, \rho^3, \rho^4\} \qquad \varphi : U_5 \to \mathbb{Z}_5$$

$$\mathbb{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} \qquad \varphi(\rho) = \bar{2} \text{ is given}$$

Since φ is a homomorphism we get that

$$\varphi(\underbrace{xy}_{\text{operation in } U_5}) = \varphi(x) \underbrace{+}_{\text{operation in } \mathbb{Z}_5} \varphi(y)$$

We get:

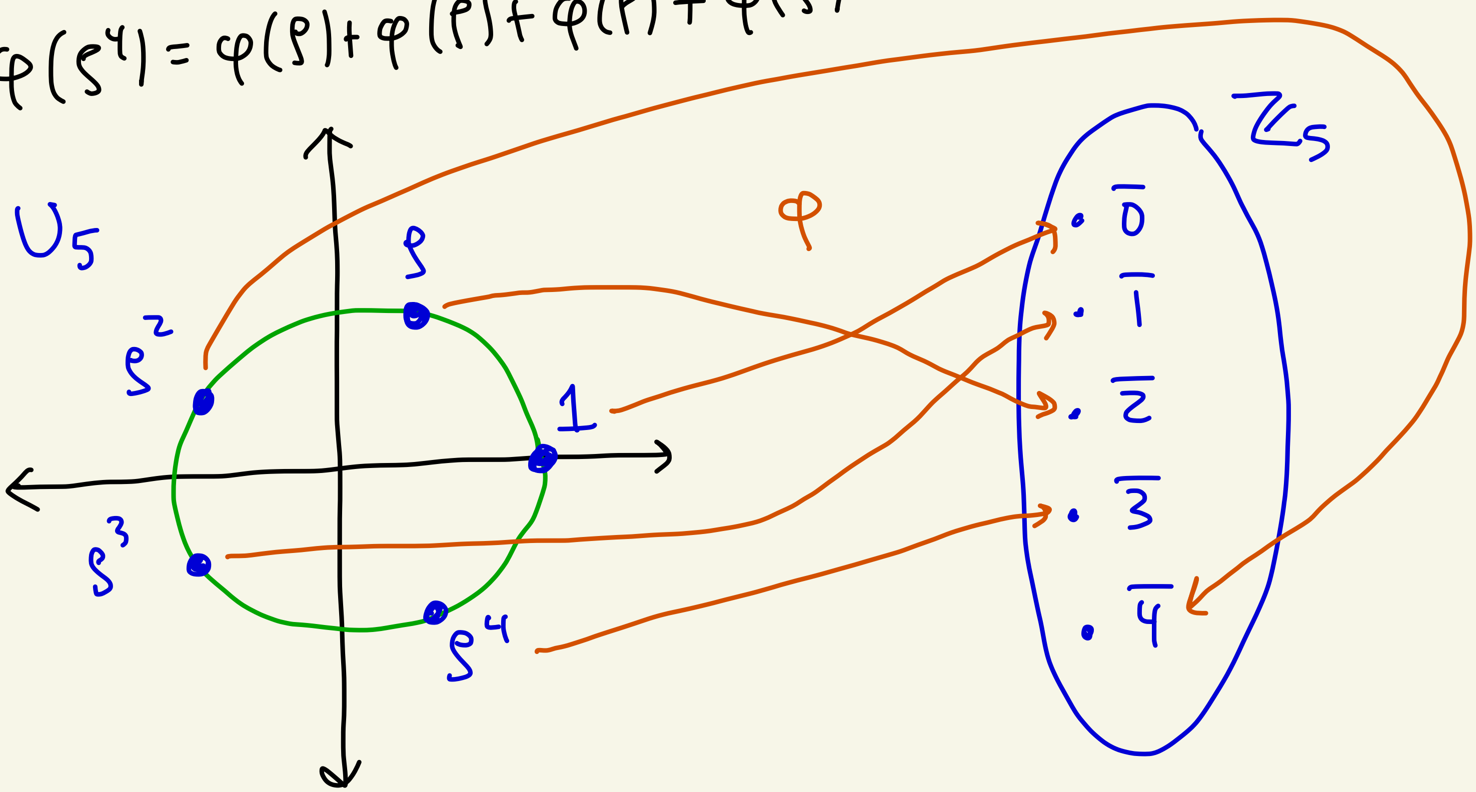
$\varphi(1) = \bar{0}$ ← identity goes to identity

$\varphi(\rho) = \bar{2}$ ← given

$\varphi(\rho^2) = \varphi(\rho\rho) = \varphi(\rho) + \varphi(\rho) = \bar{2} + \bar{2} = \bar{4}$

$\varphi(\rho^3) = \varphi(\rho\rho\rho) = \varphi(\rho) + \varphi(\rho) + \varphi(\rho) = \bar{2} + \bar{2} + \bar{2} = \bar{6} = \bar{1}$

$\varphi(\rho^4) = \varphi(\rho) + \varphi(\rho) + \varphi(\rho) + \varphi(\rho) = \bar{2} + \bar{2} + \bar{2} + \bar{2} = \bar{8} = \bar{3}$



④

$\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ is generated by $\bar{1}$ which has order 4.

Let's find the elements of $\mathbb{Z}_6$ that have orders that divide 4.

element in $\mathbb{Z}_6$	order
$\bar{0}$	1
$\bar{1}$	6
$\bar{2}$	3
$\bar{3}$	2
$\bar{4}$	3
$\bar{5}$	6

$\bar{0}$ and $\bar{3}$ have orders that divide 4

Thus a homomorphism $\varphi: \mathbb{Z}_4 \to \mathbb{Z}_6$ has two possibilities:

case 1: $\varphi(\bar{1}) = \bar{0}$

case 2: $\varphi(\bar{1}) = \bar{3}$

Let's draw these out.

Use: $\varphi(x+y) = \varphi(x) + \varphi(y)$

operation in $\mathbb{Z}_4$

operation in $\mathbb{Z}_6$

Case 1: $\varphi(\bar{1}) = \bar{0}$

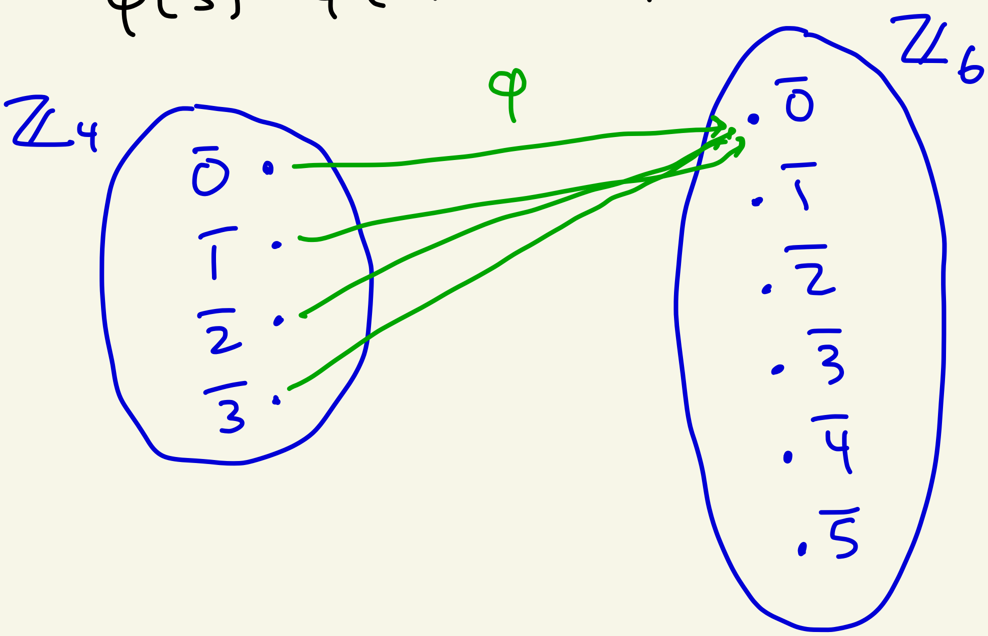
Then:

$$\varphi(\bar{0}) = \bar{0}$$ ← identity goes to identity

$$\varphi(\bar{1}) = \bar{0}$$

$$\varphi(\bar{2}) = \varphi(\bar{1}+\bar{1}) = \varphi(\bar{1}) + \varphi(\bar{1}) = \bar{0} + \bar{0} = \bar{0}$$

$$\varphi(\bar{3}) = \varphi(\bar{1}+\bar{1}+\bar{1}) = \varphi(\bar{1}) + \varphi(\bar{1}) + \varphi(\bar{1}) = \bar{0}+\bar{0}+\bar{0} = \bar{0}$$



Case 2: $\varphi(\bar{1}) = \bar{3}$

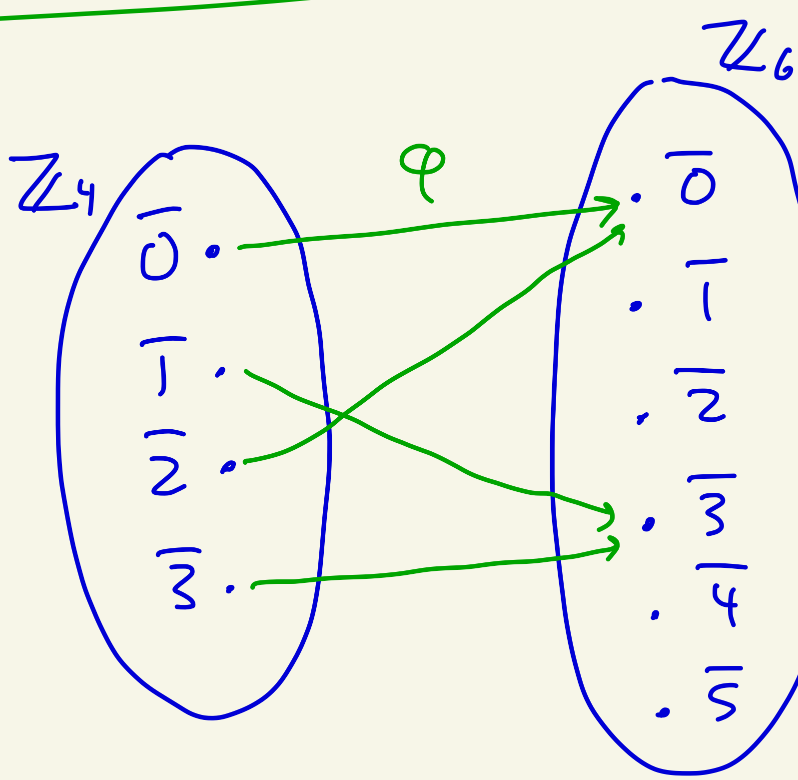
Then:

$$\varphi(\bar{0}) = \bar{0}$$ ← identity goes to identity

$$\varphi(\bar{1}) = \bar{3}$$

$$\varphi(\bar{2}) = \varphi(\bar{1}) + \varphi(\bar{1}) = \bar{3} + \bar{3} = \bar{6} = \bar{0}$$

$$\varphi(\bar{3}) = \varphi(\bar{1}) + \varphi(\bar{1}) + \varphi(\bar{1}) = \bar{3} + \bar{3} + \bar{3} = \bar{9} = \bar{3}$$



$\boxed{5}$ $U_3 = \{1, \rho, \rho^2\}$ is generated by ρ of order 3.

Let's find the elements of $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ of order dividing 3.

element of $\mathbb{Z}_6$	order
$\bar{0}$	1
$\bar{1}$	6
$\bar{2}$	3
$\bar{3}$	2
$\bar{4}$	3
$\bar{5}$	6

the elements $\bar{0}, \bar{2}, \bar{4}$ have orders that divide 3.

Thus a homomorphism $\varphi : U_3 \to \mathbb{Z}_6$ has three possibilities:

<u>Case 1:</u> $\varphi(\rho) = \bar{0}$

<u>case 2:</u> $\varphi(\rho) = \bar{2}$

<u>case 3:</u> $\varphi(\rho) = \bar{4}$

Let's draw them out.

Use: $\varphi(xy) = \varphi(x) + \varphi(y)$

operation in U_3

operation in $\mathbb{Z}_6$

Case 1: $\varphi(\rho) = \bar{0}$

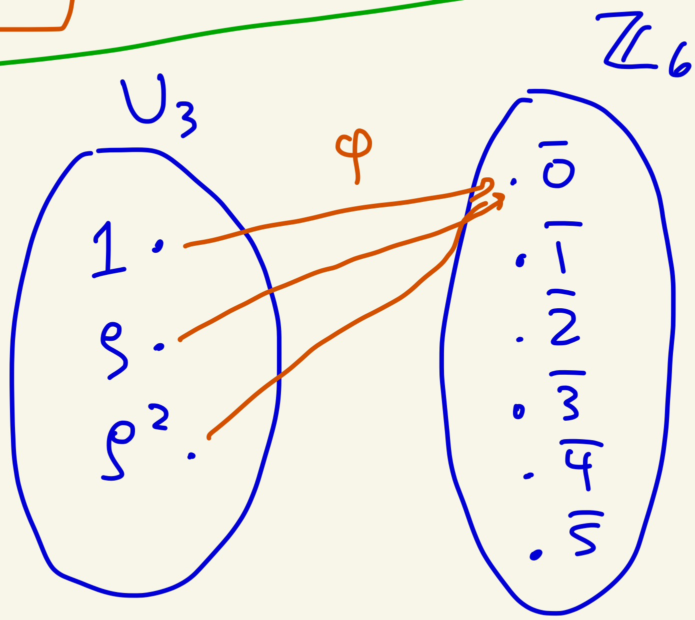
Then:

$\varphi(1) = \bar{0} \leftarrow$ $\boxed{\text{identity goes to identity}}$

$\varphi(\rho) = \bar{0}$

$\varphi(\rho^2) = \varphi(\rho) + \varphi(\rho)$
$= \bar{0} + \bar{0} = \bar{0}$

$\varphi(\rho^3) = \varphi(\rho) + \varphi(\rho) + \varphi(\rho)$
$= \bar{0} + \bar{0} + \bar{0} = \bar{0}$



Case 2: $\varphi(\rho) = \bar{2}$

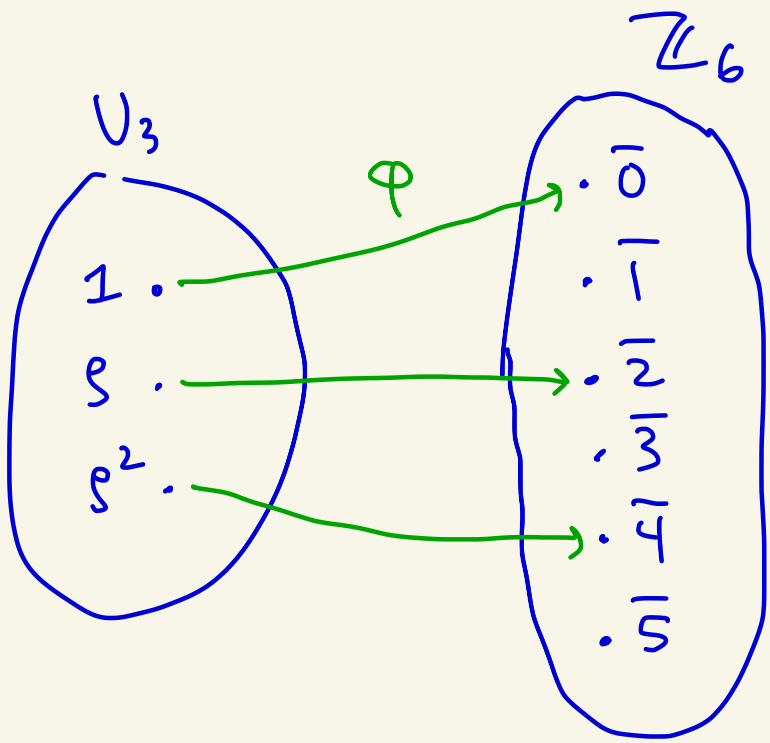
Then:

$\varphi(1) = \bar{0} \leftarrow$ $\boxed{\text{identity goes to identity}}$

$\varphi(\rho) = \bar{2}$

$\varphi(\rho^2) = \varphi(\rho) + \varphi(\rho)$
$= \bar{2} + \bar{2} = \bar{4}$

$\varphi(\rho^3) = \varphi(\rho) + \varphi(\rho) + \varphi(\rho)$
$= \bar{2} + \bar{2} + \bar{2}$
$= \bar{0}$



Case 3: $\varphi(\rho) = \overline{4}$

Then:

$\varphi(1) = \overline{0}$ ← identity goes to identity

$\varphi(\rho) = \overline{4}$

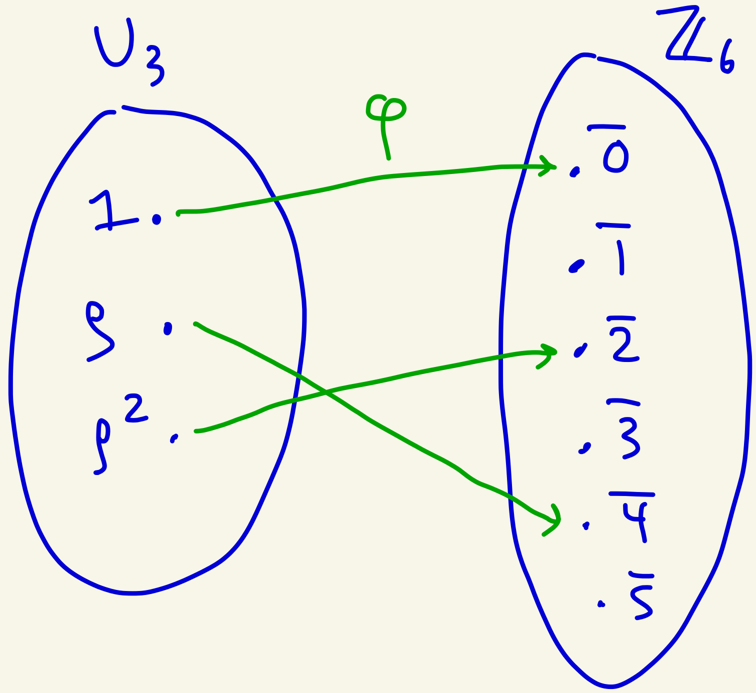
$\varphi(\rho^2) = \varphi(\rho) + \varphi(\rho)$

$= \overline{4} + \overline{4} = \overline{2}$

$\varphi(\rho^3) = \varphi(\rho) + \varphi(\rho) + \varphi(\rho)$

$= \overline{4} + \overline{4} + \overline{4}$

$= \overline{12} = \overline{0}$



⑥ $U_5 = \{1, \rho, \rho^2, \rho^3, \rho^4\}$ is generated by ρ of order 5.

Let's find the elements of $\mathbb{Z}_4$ that have order dividing 5.

element of $\mathbb{Z}_4$	order
$\bar{0}$	1
$\bar{1}$	4
$\bar{2}$	2
$\bar{3}$	4

only $\bar{0}$ has order diving 5

Thus the only homomorphism $\varphi: U_5 \to \mathbb{Z}_4$ must satisfy $\varphi(\rho) = \bar{0}$.

Then we get:

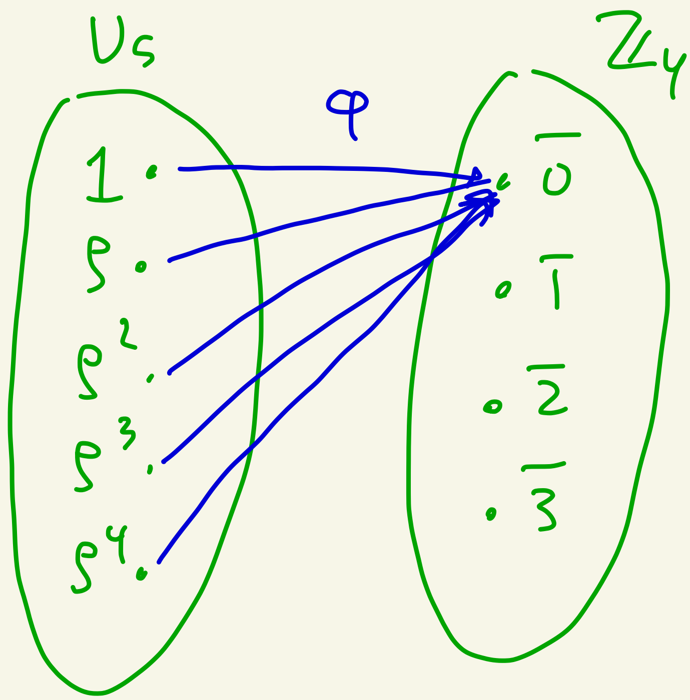
$\varphi(1) = \bar{0}$

$\varphi(\rho) = \bar{0}$

$$\begin{aligned}\varphi(\rho^2) &= \varphi(\rho) + \varphi(\rho) \\ &= \bar{0} + \bar{0} \\ &= \bar{0}\end{aligned}$$

$$\begin{aligned}\varphi(\rho^3) &= \varphi(\rho) + \varphi(\rho) + \varphi(\rho) \\ &= \bar{0} + \bar{0} + \bar{0} \\ &= \bar{0}\end{aligned}$$

$$\begin{aligned}\varphi(\rho^4) &= \varphi(\rho) + \varphi(\rho) + \varphi(\rho) + \varphi(\rho) \\ &= \bar{0} + \bar{0} + \bar{0} + \bar{0} \\ &= \bar{0}\end{aligned}$$



7

$\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ is generated by $\bar{1}$ which has order 6.

Let's find the elements of $\mathbb{Z}_3$ that have orders that divide 6.

element in $\mathbb{Z}_3$	order
$\bar{0}$	1
$\bar{1}$	3
$\bar{2}$	3

$\bar{0}$, $\bar{1}$, and $\bar{2}$ have orders that divide 6.

Thus a homomorphism $\varphi : \mathbb{Z}_6 \to \mathbb{Z}_3$ has three possibilities:

<u>case 1:</u> $\varphi(\bar{1}) = \bar{0}$

<u>case 2:</u> $\varphi(\bar{1}) = \bar{1}$

<u>case 3:</u> $\varphi(\bar{1}) = \bar{2}$

Let's draw these out.

<u>Use:</u> $\varphi(x+y) = \varphi(x) + \varphi(y)$

operation in $\mathbb{Z}_6$

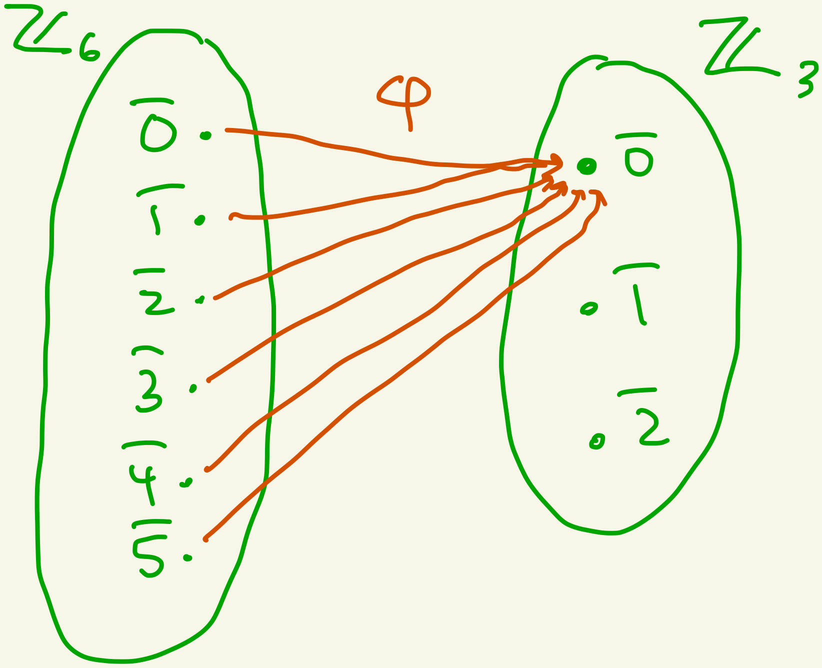
operation in $\mathbb{Z}_3$

Case 1: $\varphi(\overline{1}) = \overline{0}$

Then: $\varphi(\overline{0}) = \overline{0}$ ← identity goes to identity

And, $\varphi(\overline{2}) = \varphi(\overline{1}+\overline{1})$
$= \varphi(\overline{1}) + \varphi(\overline{1})$
$= \overline{0} + \overline{0}$
$= \overline{0}$

and so on, every element will get mapped to $\overline{0}$



Case 2: $\varphi(\overline{1}) = \overline{1}$

Then:

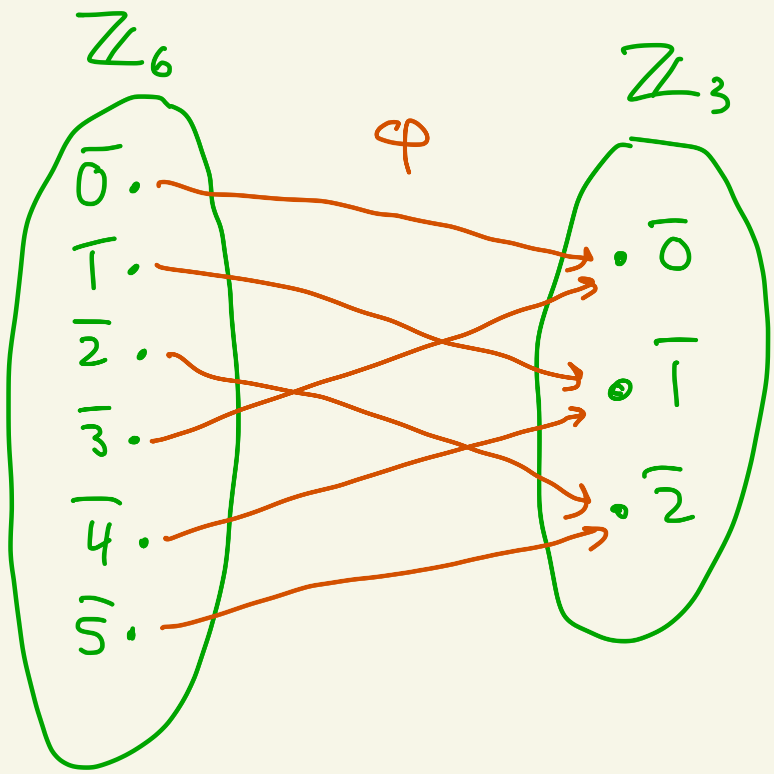
$\varphi(\overline{0}) = \overline{0}$ ← identity goes to identity

$\varphi(\overline{2}) = \varphi(\overline{1}+\overline{1})$
$= \varphi(\overline{1}) + \varphi(\overline{1})$
$= \overline{1} + \overline{1} = \overline{2}$

$\varphi(\overline{3}) = \varphi(\overline{1}) + \varphi(\overline{1}) + \varphi(\overline{1})$
$= \overline{1} + \overline{1} + \overline{1} = \overline{0}$

Similarly,
$\varphi(\overline{4}) = \overline{1}+\overline{1}+\overline{1}+\overline{1}$
$= \overline{4} = \overline{1}$

$\varphi(\overline{5}) = \overline{1}+\overline{1}+\overline{1}+\overline{1}+\overline{1}$
$= \overline{5} = \overline{2}$



Case 3: $\varphi(\bar{1}) = \bar{2}$

Then:

$\varphi(\bar{0}) = \bar{0}$ ← identity goes to identity

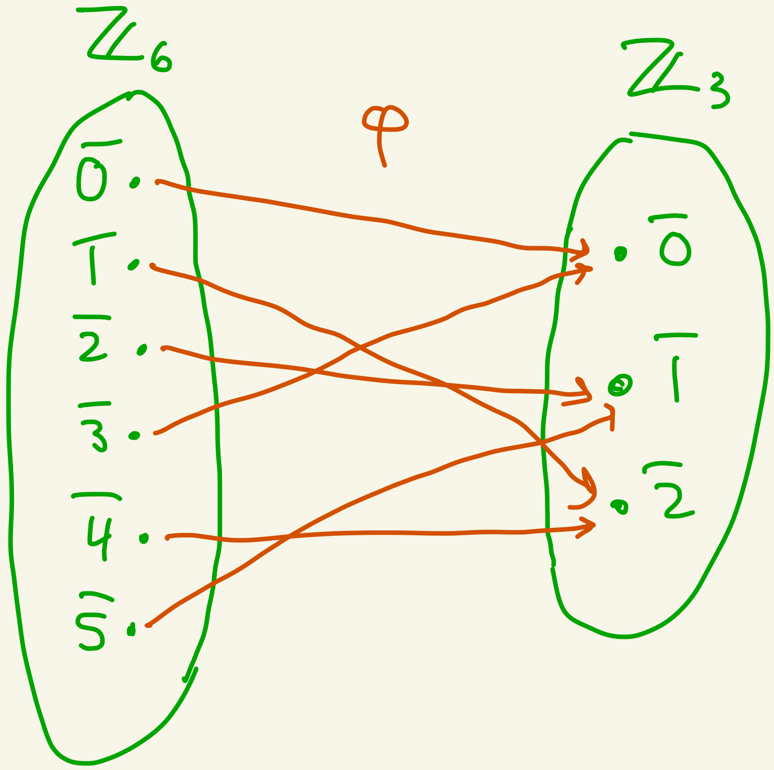
$$\varphi(\bar{2}) = \varphi(\bar{1}+\bar{1}) = \varphi(\bar{1}) + \varphi(\bar{1}) = \bar{2}+\bar{2} = \bar{4} = \bar{1}$$

$$\varphi(\bar{3}) = \varphi(\bar{1})+\varphi(\bar{1})+\varphi(\bar{1}) = \bar{2}+\bar{2}+\bar{2} = \bar{6} = \bar{0}$$

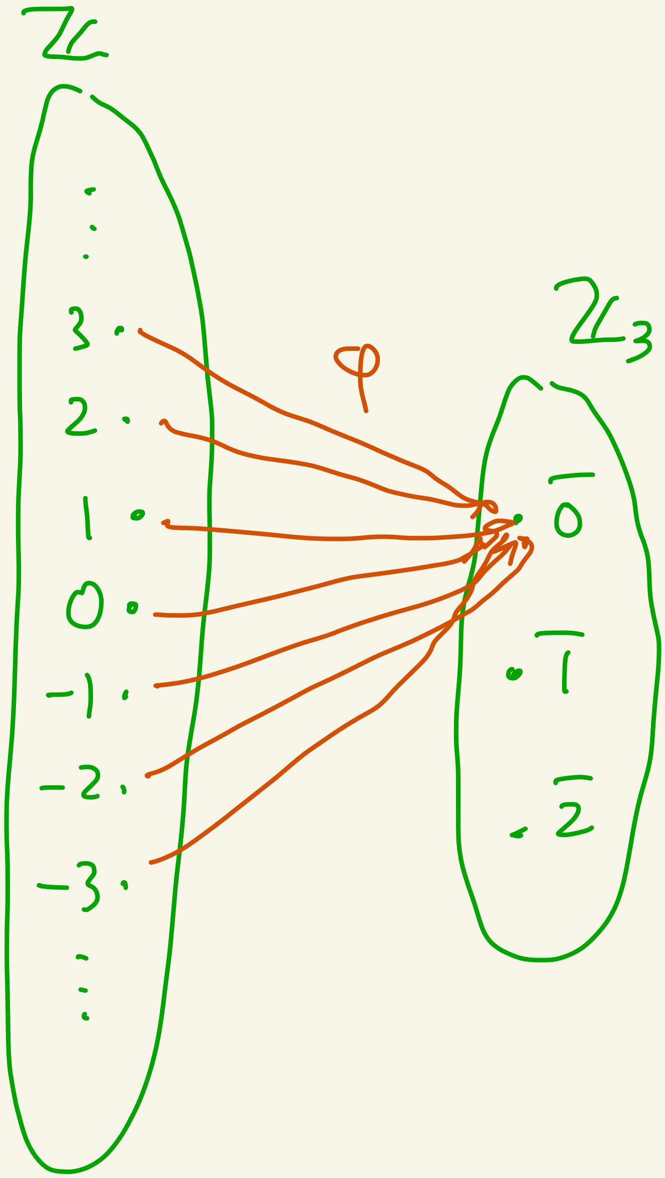
Similarly,

$$\varphi(\bar{4}) = \bar{2}+\bar{2}+\bar{2}+\bar{2} = \bar{8} = \bar{2}$$

$$\varphi(\bar{5}) = \bar{2}+\bar{2}+\bar{2}+\bar{2}+\bar{2} = \overline{10} = \bar{1}$$



⑧ $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
is an infinite cyclic group. 1 is a generator.
We can send $\bar{1}$ to any element of $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$.



case 1: $\varphi(1) = \bar{0}$

Then:

$$\varphi(0) = \bar{0}$$ identity goes to identity

$$\varphi(1) = \bar{0}$$

$$\begin{aligned} \varphi(2) &= \varphi(1+1) \\ &= \varphi(1) + \varphi(1) \\ &= \bar{0} + \bar{0} \\ &= \bar{0} \end{aligned}$$

$$\begin{aligned} \varphi(3) &= \varphi(1) + \varphi(1) + \varphi(1) \\ &= \bar{0} + \bar{0} + \bar{0} = \bar{0} \end{aligned}$$

Similarly, $\varphi(n) = \bar{0}$ if $n > 0$

$$\begin{aligned} \varphi(-1) &= [\varphi(1)]^{-1} \\ &= \bar{0}^{-1} = \bar{0} \end{aligned}$$

$$\begin{aligned} \varphi(-2) &= \varphi(-1) + \varphi(-1) \\ &= \bar{0} + \bar{0} = \bar{0} \end{aligned}$$

Similarly $\varphi(n) = \bar{0}$ if $n < 0$.

So, $\varphi(n) = \bar{0}$ for all n.

Case 2: $\varphi(1)=\bar{1}$

Then:

$\varphi(0)=\bar{0}$ ← identity goes to identity

$\varphi(1)=\bar{1}$

$$\begin{aligned}\varphi(2)&=\varphi(1+1)\\&=\varphi(1)+\varphi(1)\\&=\bar{1}+\bar{1}=\bar{2}\end{aligned}$$

$$\begin{aligned}\varphi(3)&=\varphi(1)+\varphi(1)+\varphi(1)\\&=\bar{1}+\bar{1}+\bar{1}=\bar{0}\end{aligned}$$

$\varphi(4)=\bar{1}$

$\varphi(5)=\bar{2}$

$\varphi(6)=\bar{0}$

and so on...

(it cycles $\bar{1},\bar{2},\bar{0},\bar{1},\bar{2},\bar{0},\ldots$)

Also,

$$\begin{aligned}\varphi(-1)&=[\varphi(1)]^{-1}\\&=\bar{1}^{-1}=\bar{2}\end{aligned}$$

$$\begin{aligned}\varphi(-2)&=\varphi(-1)+\varphi(-1)\\&=\bar{2}+\bar{2}=\bar{1}\end{aligned}$$

$\varphi(-3)=\bar{2}+\bar{2}+\bar{2}=\bar{0}$

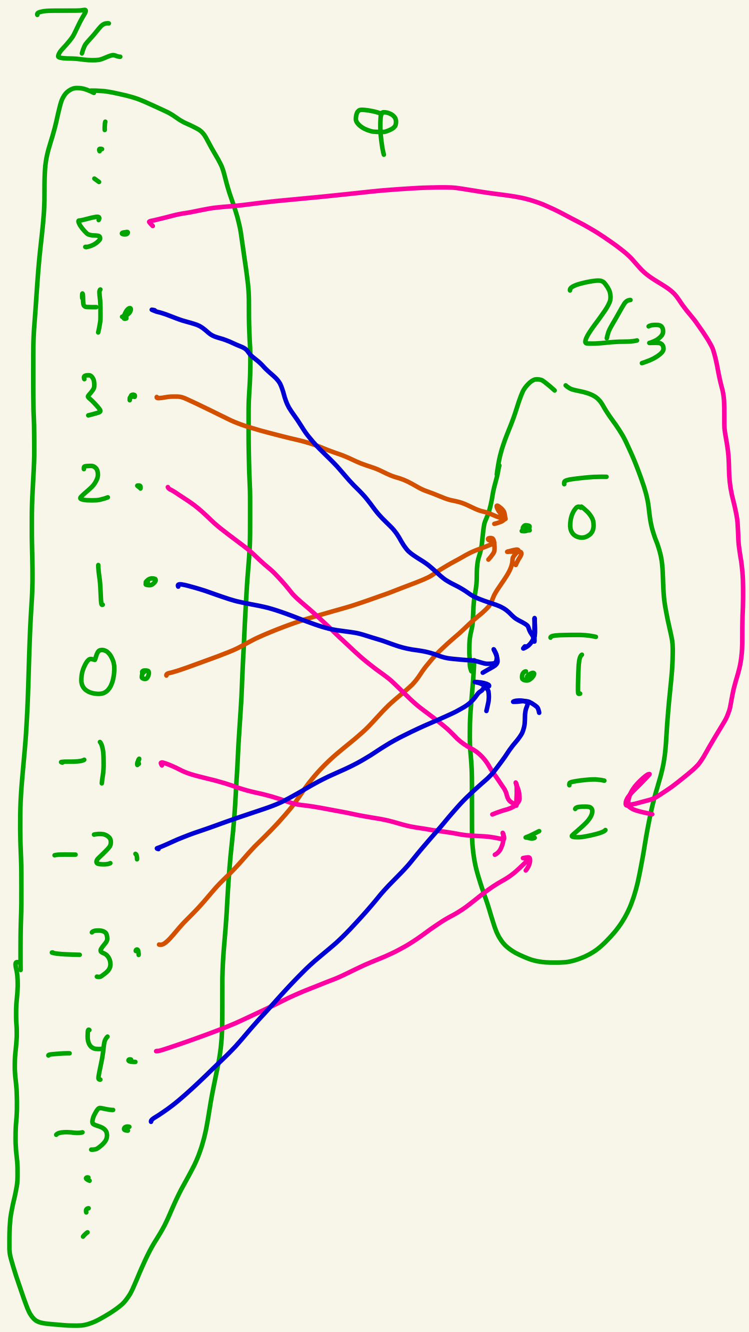
$\varphi(-4)=\bar{2}$

$\varphi(-5)=\bar{1}$

$\varphi(-6)=\bar{0}$

and so on...

(it cycles $\bar{2},\bar{1},\bar{0},\bar{2},\bar{1},\bar{0},\ldots$)



Case 3: $\varphi(1)=\bar{2}$

Then:

$\varphi(0)=\bar{0}$ ← identity goes to identity

$\varphi(1)=\bar{2}$

$$\varphi(2)=\varphi(1+1) = \varphi(1)+\varphi(1) = \bar{2}+\bar{2}=\bar{4}=\bar{1}$$

$$\varphi(3)=\varphi(1)+\varphi(1)+\varphi(1) = \bar{2}+\bar{2}+\bar{2}=\bar{6}=\bar{0}$$

$\varphi(4)=\bar{2}$

$\varphi(5)=\bar{1}$

$\varphi(6)=\bar{0}$

and so on...

(it cycles $\bar{0},\bar{2},\bar{1},\bar{0},\bar{2},\bar{1},\dots$)

Also,

$$\varphi(-1)=[\varphi(1)]^{-1} = \bar{2}^{-1}=\bar{1}$$

$$\varphi(-2)=\varphi(-1)+\varphi(-1) = \bar{1}+\bar{1}=\bar{2}$$

$\varphi(-3)=\bar{1}+\bar{1}+\bar{1}=\bar{0}$

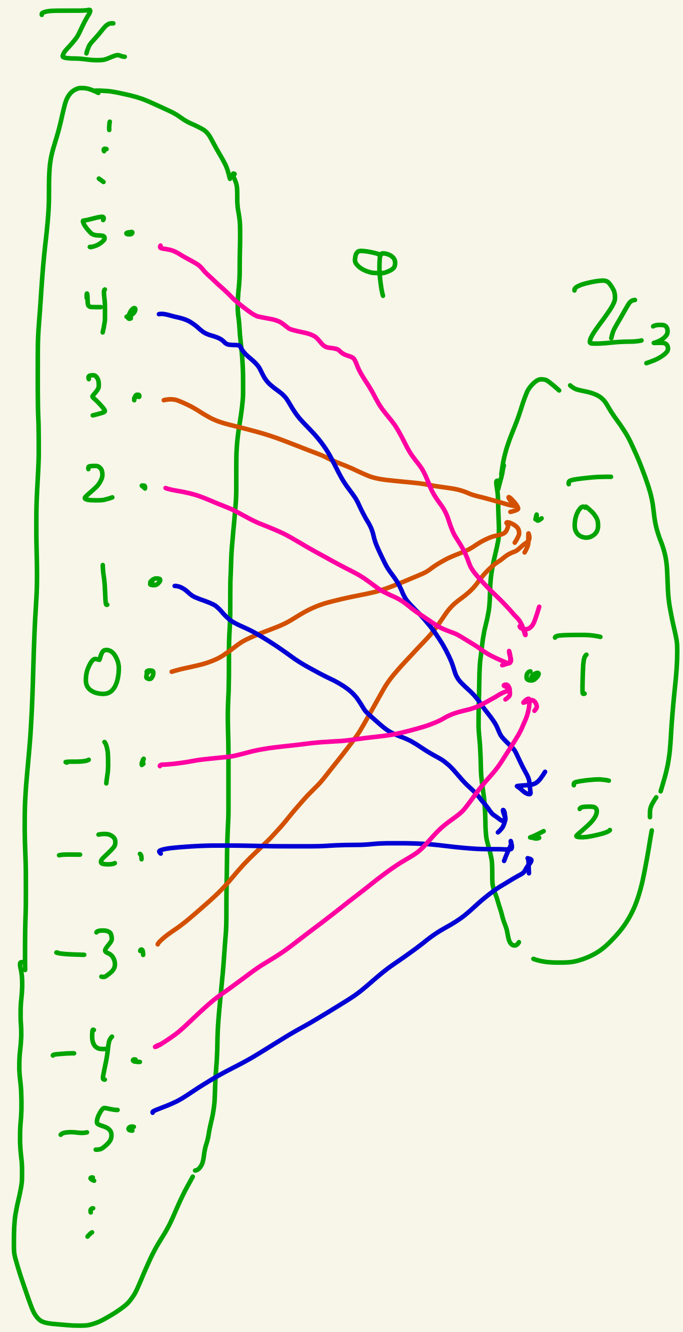
$\varphi(-4)=\bar{1}$

$\varphi(-5)=\bar{2}$

$\varphi(-6)=\bar{0}$

and so on...

(it cycles $\bar{1},\bar{2},\bar{0},\bar{1},\bar{2},\bar{0},\dots$)



9

Suppose $\varphi: \mathbb{Z} \to \mathbb{Z}$ is a homomorphism with $\varphi(1) = 5$.

Since φ is a homomorphism we know that $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x, y \in \mathbb{Z}$.

Then:

$\varphi(0) = 0 \leftarrow$ identity goes to identity

$\varphi(1) = 5$

$\varphi(2) = \varphi(1+1) = \varphi(1) + \varphi(1) = 5+5 = 10$

$\varphi(3) = \varphi(1+1+1) = \varphi(1) + \varphi(1) + \varphi(1) = 5+5+5 = 15$

$\vdots \qquad \vdots$

and so on, if $n > 0$ then

$$\varphi(n) = \underbrace{\varphi(1) + \varphi(1) + \cdots + \varphi(1)}_{n \text{ times}} = \underbrace{5+5+\cdots+5}_{n \text{ times}} = 5n$$

$$\varphi(-1) = \underbrace{\varphi(1)^{-1}}_{\substack{\text{here I mean} \\ \text{inverse in the additive group } \mathbb{Z}}} = 5^{-1} = -5$$

$\varphi(-2) = \varphi(-1) + \varphi(-1) = -5-5 = -10$

$\vdots \qquad \vdots$

and so on, if $n < 0$ then

$$\varphi(n) = \underbrace{\varphi(-1) + \varphi(-1) + \cdots + \varphi(-1)}_{-n \text{ times}} = \underbrace{-5-5-\cdots-5}_{-n \text{ times}} = 5n$$

$\boxed{\text{Thus, } \varphi(n) = 5n \text{ for all } n \in \mathbb{Z}.}$

10 Let e be the identity of G.

(a) Let $x \in G$. Then,

$$\begin{aligned} \langle x \rangle &= \{x^n \mid n \in \mathbb{Z}\} \\ &= \{\ldots, x^{-3}, x^{-2}, x^{-1}, e, x, x^2, x^3, \ldots\} \\ &= \{\ldots, (x^{-1})^3, (x^{-1})^2, x^{-1}, e, (x^{-1})^{-1}, (x^{-1})^{-2}, (x^{-1})^{-3}, \ldots\} \\ &= \{(x^{-1})^n \mid n \in \mathbb{Z}\} \\ &= \langle x^{-1} \rangle. \end{aligned}$$

(b) This follows from part (a).

If x has infinite order then $\langle x \rangle$ is infinite and then so is $\langle x^{-1} \rangle$.

Thus x^{-1} has infinite order.

If x has finite order n, then

$$n = |\langle x \rangle| = |\langle x^{-1} \rangle|$$

from class

part (a)

So, x^{-1} has order n also.

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11 (a)

Let $\frac{m}{n} \in \mathbb{Q}$ with $m, n \in \mathbb{Z}$, $n \neq 0$.

We will show that $\frac{m}{n}$ cannot generate $\mathbb{Q}$.

If $\frac{m}{n} = 0$, then $\langle \frac{m}{n} \rangle = \{0\}$.

So in this case $\frac{m}{n}$ does not generate $\mathbb{Q}$.

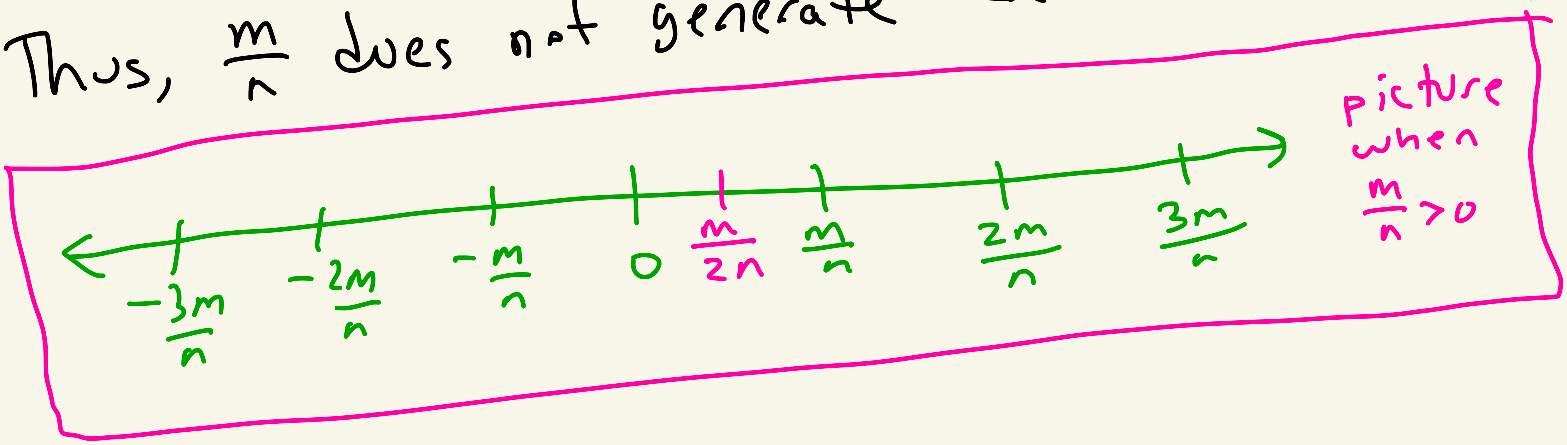
Suppose $\frac{m}{n} \neq 0$.

Then,

$$\langle \tfrac{m}{n} \rangle = \left\{ \ldots, -\frac{3m}{n}, -\frac{2m}{n}, -\frac{m}{n}, 0, \frac{m}{n}, \frac{2m}{n}, \frac{3m}{n}, \ldots \right\}$$

Note that $\frac{m}{2n} \in \mathbb{Q}$ but $\frac{m}{2n} \notin \langle \frac{m}{n} \rangle$.

Thus, $\frac{m}{n}$ does not generate $\mathbb{Q}$.



Thus, in either case $\frac{m}{n}$ does not generate $\mathbb{Q}$

So, $\mathbb{Q}$ is not cyclic.

11(b) (Method 1)

Suppose $\mathbb{R}$ is cyclic.

Then from class, since $\mathbb{R}$ is an infinite cyclic group, there exists an isomorphism $\varphi : \mathbb{Z} \to \mathbb{R}$.

But then φ is a bijection (one-to-one and onto) between $\mathbb{Z}$ and $\mathbb{R}$.

However, from Math 3450 we know that $\mathbb{Z}$ is countable and $\mathbb{R}$ is not countable.

That is, no such bijection exists.

Hence $\mathbb{R}$ is not cyclic. $\blacksquare$

Method 2: Do a similar proof to what I wrote for 11(a). $\blacksquare$

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12 (a)

$\varphi_3 : \mathbb{Z} \to \mathbb{Z}_3$

$\varphi_3(x) = \overline{x}$

Thus,

$\varphi(0) = \overline{0}$

$\varphi(1) = \overline{1}$

$\varphi(2) = \overline{2}$

$\varphi(3) = \overline{3} = \overline{0}$

$\varphi(4) = \overline{4} = \overline{1}$

$\varphi(5) = \overline{5} = \overline{2}$

and so on...

$\varphi(-1) = \overline{-1} = \overline{2}$

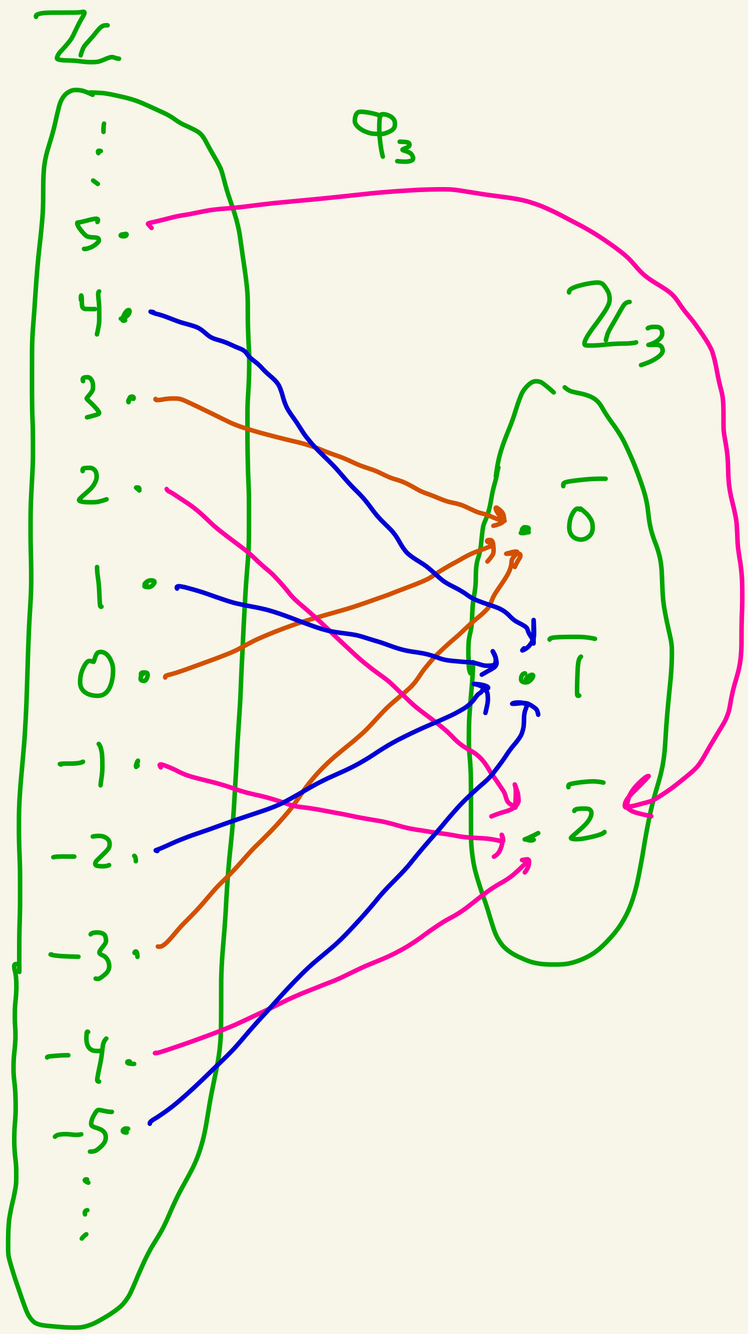
$\varphi(-2) = \overline{-2} = \overline{1}$

$\varphi(-3) = \overline{-3} = \overline{0}$

$\varphi(-4) = \overline{-4} = \overline{2}$

$\varphi(-5) = \overline{-5} = \overline{1}$

and so on...



12 (b)

Given $x, y \in \mathbb{Z}$ we have

$$\varphi_n(x+y) = \overline{x+y} = \overline{x} + \overline{y} = \varphi_n(x) + \varphi_n(y)$$

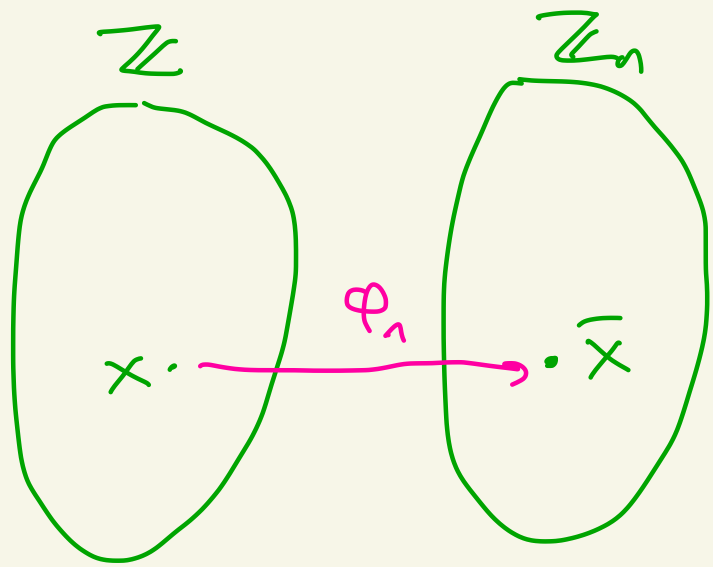
def of φ_n

operation in $\mathbb{Z}_n$

Thus, φ_n is a homomorphism.

---

12 (c)

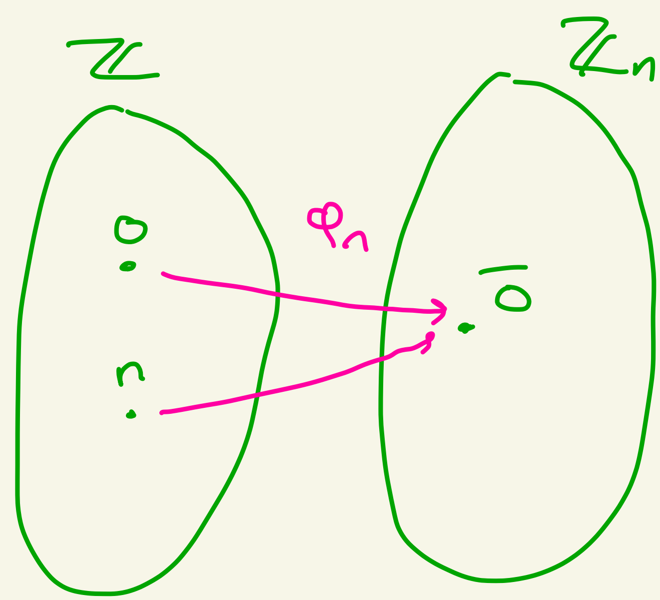


φ_n is onto:

Let $\overline{x} \in \mathbb{Z}_n$.
Then $x \in \mathbb{Z}$ and

$$\varphi_n(x) = \overline{x}.$$

Thus, φ_n is onto.



φ_n is not one-to-one:

Note that

$$\varphi_n(0) = \overline{0}$$

and

$$\varphi_n(n) = \overline{n} = \overline{0}$$

Thus, $\varphi_n(0) = \varphi_n(n)$ but $0 \neq n$.
So, φ_n is not one-to-one.

12 (d)

We have that

$$\begin{aligned}
\ker(\varphi_n) &= \{ x \mid x \in \mathbb{Z} \text{ and } \varphi_n(x) = \bar{0} \} \\
&= \{ x \mid x \in \mathbb{Z} \text{ and } \bar{x} = \bar{0} \} \\
&= \{ x \mid x \in \mathbb{Z} \text{ and } x \equiv 0 \pmod{n} \} \\
&= \{ x \mid x \in \mathbb{Z} \text{ and } n \mid (x - 0) \} \\
&= \{ x \mid x \in \mathbb{Z} \text{ and } n \mid x \} \\
&= \{ x \mid x \in \mathbb{Z} \text{ and } x = nk \text{ for some } k \in \mathbb{Z} \} \\
&= \{ nk \mid k \in \mathbb{Z} \} \\
&= n\mathbb{Z}
\end{aligned}$$

$\boxed{//}$